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Inexhaustible homogeneous structures

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Abstract

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We discuss the notions of inexhaustible, weakly inexhaustible, age-inexhaustible and closure-inexhaustible for countable homogeneous structures. Then we characterize these properties of homogeneous structures in various ways. We also exhibit several examples which show that the assumption of finite language is necessary for some of the results.

1. Introduction

Throughout the paper $\mathscr{L}$ is a first-order relational language. A countable infinite model H of $\mathscr{L}$ is *homogeneous* if, for all finite subsets $A, B \subseteq H$ and isomorphisms $f: A \to B$, f can be extended to an automorphism of H [3]. The letter H will always denote a homogeneous structure, $G = \mathrm{Aut}(H)$. We say $K \subseteq H$ *is an orbit over a finite set* if there is some finite set $A \subseteq H$ such that $A \cap K = \emptyset$ and K is an orbit of G_A. Here G_A is the pointwise stabilizer of A. It is known [2] and not difficult to see that, for a homogeneous structure H, the following two statements are equivalent:

(i) For all finite $A \subseteq H$, $H \setminus A$ is isomorphic to H.

(ii) For every finite $A \subseteq H$, every orbit of G_A is infinite.

Homogeneous structures satisfying (i) will be called *inexhaustible* and have been considered in [7] under the name of perfectly homogeneous structures. The corresponding equivalent 'amalgamation condition' is called strong amalgamation by Fraissé [3] and condition IV by Jonsson [4].

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In this paper we examine homogeneous structures which are exhaustible and, so, do not satisfy statement (i). For an infinite model M of $\mathscr{L}$, *age* M is the set of finite substructures (restrictions) of M up to an isomorphism. If the language $\mathscr{L}$ is infinite, another notion of *age* has also been investigated by Pouzet [8–10]. In this other notion the age consists of the finite restrictions of the finite reducts of the relational structure. Note that in our definition an element of *ageM* might have infinitely many relation symbols. We say that H is *age-inexhaustible* if, for all finite $A \subseteq M$, $ageM = age(M - A)$ holds. The structure H is *weakly inexhaustible* if for all finite $A \subseteq H$ there exists an $\bar{H} \subseteq H - A$ such that $\bar{H}$ is isomorphic to H. Pouzet [8, 9] initiated the study of weakly inexhaustible structures, which he calls inexhaustible structures. We should also mention that the notion of inexhaustible relational structure and the notion of age-inexhaustible is closely related to the notion of kernel investigated by Pouzet [10] and the notion of core investigated by Kuekker [10]. So, for example, in the case that the language $\mathscr{L}$ is finite, Theorem 1.1 follows readily from Pouzets result that then the kernel of H is the union of its finite 1-element orbits [10, p. 372]. Also, with some more effort, all of Theorem 1.1 can be deduced from the results in [10]. We will give a self-contained argument.

Clearly, if H is inexhaustible then H is weakly inexhaustible and if H is weakly inexhaustible then H is age-inexhaustible. Our first theorem characterizes the age-inexhaustible structures.

Theorem 1.1 (Pouzet [8–10]; the language is finite). *The homogeneous structure H is age-inexhaustible iff all orbits of the empty set are infinite.*

The proof of the theorem is based on the following lemma.

Lemma 1.2. *Let H be a homogeneous structure with automorphism group G and A, B be two finite disjoint subsets of H such that every element of B belongs to an infinite orbit of A. Then there are automorphisms $f_n \in G_A$ such that, for all $n, m \in \omega$, with $n \neq m$, $f_n(B) \cap f_m(B) = \emptyset$.*

Proof. If $|B| = 1$, the conclusion of the lemma follows trivially. Hence, it is enough to prove that if there are such automorphisms f_n for B and $b \in H - (A \cup B)$ is in some infinite orbit K of A then there are automorphisms $g_n \in G_A$ such that, for all $n, m \in \omega$, with $n \neq m$, $g_n(B \cup \{b\}) \cap g_m(B \cup \{b\}) = \emptyset$.

If no subsequence of the sequence $\langle f_n \rangle$ has the required properties, then there is an infinite subsequence $\langle h_n \rangle$ of $\langle f_n \rangle$ such that, for all $n, m \in \omega$, $h_n(b) = h_m(b) := \bar{b}$. Since $h_0 \in G_A$, it follows that $\bar{b} \in K$. Hence, there are automorphisms $l_n \in G_A$ such that for all $n, m \in \omega$ and $n \neq m$, $l_n(\bar{b}) \neq l_m(\bar{b})$.

We are now in a position to define the sequence $\langle g_n \rangle$. Put $g_0 = h_0$ and assume that $g_0, g_1, \ldots, g_n$ have been determined. Let r, s be numbers such that $l_r(h_s(A \cup B \cup \{b\})) \cap \bigcup_{i=1}^{n} g_i(A \cup B \cup \{b\}) = \emptyset$. Such numbers r, s exist because $l_i(\bar{b}) \in \bigcup_{i=1}^{n} g_i(A \cup B \cup \{b\})$

for at most finitely many indices i and $l_r(h_n(B)) \cap l_r(h_m(B)) = \emptyset$ for $n \neq m$ since l_r is an automorphism. Put $g_{n+1} = l_r \circ h_s$. $\square$

Proof of Theorem 1.1. If $K \subseteq H$ is a finite orbit of H then $H \setminus K$ cannot contain a copy of K. Now assume that every orbit of the empty set is infinite, $X \subseteq H$ is finite and $B \in ageH$. Using Lemma 1.2 for $A = \emptyset$, we can find infinitely many pairwise disjoint copies of B in H. As X is finite, some of these copies are subsets of $H \setminus X$. This proves the theorem. $\square$

A weakly inexhaustible structure is age-inexhaustible. We show that for finite languages the two properties are equivalent, which is not necessarily the case for infinite languages (Example 1.4). Our proof of Theorem 1.3 is self-contained. As a referee of this paper has pointed out, Theorem 1.3 is due to Pouzet and can be easily obtained using the following argument. If a homogeneous $\mathscr{L}$-structure H is age-inexhaustible then, according to [1, 5.5, p. 294], there exists an $\mathscr{L}$-structure H' with $ageH' = ageH$ and H' is weakly inexhaustible. Because $ageH' = ageH$, H' can be embedded into H [1, 1.3(2), p. 314]. But H then is clearly weakly age-inexhaustible.

Theorem 1.3 (Pouzet [8–10]. *If H is a homogeneous structure with finite language $\mathscr{L}$ and every orbit of the empty set is infinite, then H is weakly inexhaustible.*

Proof. Assume that the base set of H is ω and that H_n is the substructure induced by n. Let $A \subseteq H$ be finite and $F_n = \{f : H_n \to (H - A) \mid f \text{ is an embedding}\}$. $F = \bigcup_\omega F_n$. It follows from Theorem 1.1 that, for all $n \in \omega$, $F_n \neq \emptyset$. We define an equivalence relation $\overset{n}{\sim}$ on F_n by $f \overset{n}{\sim} g$ iff $(f \circ g^{-1}) \cup \mathrm{id}_A$ is an isomorphism.

Because $\mathscr{L}$ is finite, there are only finitely many equivalence classes of $\overset{n}{\sim}$. We put $\sim = \bigcup_\omega \overset{n}{\sim}$. For $f, g \in F$, put $g \leqslant f$ iff $(f \circ g^{-1}) \cup \mathrm{id}_A$ is an embedding. If $f' \sim f$, $g' \sim g$ and $g \leqslant f$, then $g' \leqslant f'$. Hence, $\leqslant$ induces a partial order P on the equivalence classes of f. The partial order P has infinitely many elements, each element has a finite cover and P is well-founded. (If $(Q, \leqslant)$ is a partial order and $x, y \in Q$ with $x < y$, then y is in the cover of x if, for all elements $z \in Q$, $x \leqslant z \leqslant y$ implies $x = z$ or $y = z$. The partial order Q has finite cover if, for every element $x \in Q$, the cover of x is a finite set.) By Königs lemma, the partial order P contains an infinite chain C which contains an element from each level of P. Let $C = C_0, C_1, C_2, \ldots$ and let $g_0 \in C_0$. Assume that we have already chosen $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots \subseteq g_{n-1}$, with $g_i \in C_i$ for all $i \in n$. For $f_n \in C_n$, we have $g_{n-1} < f_n$ and, hence, $h = (f_n \circ g_{n-1}^{-1}) \cup \mathrm{id}_A$ is an isomorphism. The function h can be extended to an automorphism l of H. We have $l^{-1} f_n \sim f_n$ because $(l^{-1} f_n) f_n^{-1} = l^{-1} \mid \mathrm{im} f_n$ is an isomorphism and $l^{-1} \mid A = \mathrm{id}_A$. Also, $g_{n-1} \subseteq l^{-1} f_n$ and, so, we put $g_n = l^{-1} f_n$. This process yields an infinite sequence $g_0 \subseteq g_1 \subseteq g_2 \subseteq \cdots$ with $g_i \in F_i$ for all $i \in \omega$. Clearly, then $g = \bigcup_\omega g_i$ is an embedding of H into $H - A$. $\square$

Next we give an example which shows that Theorem 1.3 may fail for an infinite language $\mathscr{L}$ even in the case when the group G is transitive.

Example 1.4. The relations of $\mathscr{L}$ are τ_n, $n\in\omega$. All the relations τ_n are binary. The elements of H are the integers, for $n\in\omega$, $\tau_n(x,y)$ holds if and only if $|x-y|=n$. This is a homogeneous structure (the translations, the reflextion, $x\to -x$, and their compositions are the only isomorphisms even between finite isomorphic substructures). The automorphism group G of H is transitive and there is no proper substructure of H isomorphic to H. From this it follows that, for all finite, nonempty $A\subseteq H$, the substructure $H-A$ does not contain a copy of H.

Definition 1.5. For finite $A\subseteq H$, $\mathrm{cl}\,A:=\{x\in H\mid x \text{ belongs to a finite orbit of } A\}\cup A$, that is, $\mathrm{cl}(A)$ is the union of finite orbits of G_A. We say the homogeneous structure H has closure if, for all finite $A\subseteq H$, $\mathrm{cl}\,A$ is finite and there is a finite $A\subseteq H$ with $\mathrm{cl}\,A\neq A$.

Observe that H has closure if H has finite language $\mathscr{L}$ and there is a finite $A\subseteq H$ with $A\neq \mathrm{cl}\,A$. If H has closure then the function cl is a closure operator in the usual sense. The only not completely trivial property is $\mathrm{cl}\,\mathrm{cl}\,A=\mathrm{cl}\,A$. But this follows from the fact that if M is a subgroup of finite index of the permutation group G and the orbit under M of the element x is finite then the orbit under G of x is also finite, as well as from the observation that the index of $G_{\mathrm{cl}\,A}$ as a subgroup of G_A is finite. If $\mathscr{L}$ is the language of graphs or directed graphs then the homogeneous models or $\mathscr{L}$ are known [1, 6]. It turns out that for these structures, for any finite $A\subseteq H$, $H-\mathrm{cl}\,A$ is isomorphic to H.

Definition 1.6. The homogeneous structure H is closure-inexhaustible if, for all finite $A\subseteq H$, H is isomorphic to $H-\mathrm{cl}\,A$.

Lemma 1.7. *The following conditions are equivalent for a homogeneous structure H:*
(i) *H is closure-inexhaustible.*
(ii) *For all finite $A,B\subseteq H$, if $\mathrm{cl}\,A\cap B=\emptyset$ then $\mathrm{cl}\,A\cap \mathrm{cl}\,B=\emptyset$.*

Proof. Observe first that a countable $\mathscr{L}$-structure M with $ageM\subseteq ageH$ is isomorphic to the countable homogeneous $\mathscr{L}$-structure H just in case the following mapping condition [3] holds:

(MC) For all $A\in ageH$ and $x\in A$, if g is an embedding from $A-\{x\}$ into M, then there exists an extension g' of g which is an embedding of A into M.

(ii) $\Rightarrow$ (i): let $A\subseteq H$ be finite. We will show that $H-\mathrm{cl}\,A$ satisfies MC. So, assume that $g: B-\{x\}\to H-\mathrm{cl}\,A$ is an embedding, and that $x\in B$. There is an extension g' of g which is an embedding from B into H. If the orbit of $g(B-\{x\})$ containing $g'(x)$ is infinite then there is a y in this orbit with $y\notin \mathrm{cl}\,A$. Then $g\cup\{(x,y)\}$ is the desired extension of g. If the orbit of $g(B-\{x\})$ containing $g'(x)$ is finite, then $g'(x)\notin \mathrm{cl}\,A$ and, hence, g' is the desired extension of g.

Not (ii) implies not (i): If not (ii), then there are two finite $A, B\subseteq H$, with $\mathrm{cl}\,A\cap B=\emptyset$ but $\mathrm{cl}\,A\cap \mathrm{cl}\,B\neq\emptyset$. The function $\mathrm{id}_{\mathrm{cl}\,B-\mathrm{cl}\,A}$ is an embedding of $\mathrm{cl}\,B-\mathrm{cl}\,A$ into $H-\mathrm{cl}\,A$. For $x\in \mathrm{cl}\,A\cap \mathrm{cl}\,B$ there is no extension f of this embedding with $f(x)\in H-\mathrm{cl}\,A$. □

Definition 1.8. The homogeneous structure H has singly generated closure if there is an $x\in H$ with $|\mathrm{cl}(\{x\})|\geqslant 2$ and, for all finite $A\subseteq H$, $\mathrm{cl}\,A=\bigcup_{x\in A}\mathrm{cl}(\{x\})$.

Clearly, H has singly generated closure if H has closure and, for all finite $A, B\subseteq H$, with $A\cap B=\emptyset$, $\mathrm{cl}(A\cup B)=\mathrm{cl}\,A\cup \mathrm{cl}\,B$. Observe also that cl is singly generated iff, for all finite $A, B\subseteq H$ and infinite orbits K of A and L of B, $K\cap L$ is empty or infinite.

Definition 1.9. A homogeneous structure H is transitive if its automorphism group G is transitive, that is, for every two elements $x, y\in H$, there is an element $g\in G$, with $g(x)=y$. $A\subseteq H$ is an imprimitivity class if A is an imprimitivity class of G. The equivalence relation $\sim$ partitions H into imprimitivity class if, for all $x, y\in H$ and $g\in G$, $x\sim y$ implies $g(x)\sim g(y)$.

Theorem 1.10. *If H is a transitive homogeneous structure with closure, then the following are equivalent:*

(i) *H is closure-inexhaustible.*

(ii) *The closure of H is singly generated.*

(iii) *H has a partition into finite pairwise homogeneous structures, each being an imprimitivity class closed in* cl *and if A is a finite subset of H such that no element of A is in the closure of any other element of A, then* $\mathrm{cl}\,A=\bigcup_{a\in A}\mathrm{cl}(a)$.

Proof. (i) $\Rightarrow$ (ii): If the closure of H is not singly generated then there exists a finite $A\subseteq H$ and $a\in \mathrm{cl}\,A$ but $a\notin\bigcup_{x\in A}\mathrm{cl}(\{x\})$. If H is closure-inexhaustible, we get then from Lemma 1.7 that, for all $x\in A$, $\mathrm{cl}(\{a\})\cap \mathrm{cl}(\{x\})=\emptyset$ and then $\mathrm{cl}(\{a\})\cap\bigcup_{x\in A}\mathrm{cl}(\{x\})=\emptyset$. Now $A\subseteq\bigcup_{x\in A}\mathrm{cl}(\{x\})$ and $A\cap \mathrm{cl}(\{a\})=\emptyset$. Again, from Lemma 1.7, $\emptyset=\mathrm{cl}\,A\cap \mathrm{cl}(\{a\})\ni a$, a contradiction.

(ii) $\Rightarrow$ (iii): Because H is transitive and the closure of every element of H is finite, it follows that the relation $\sim$ given by $a\sim b$ iff $a\in \mathrm{cl}(\{b\})$ is an equivalence relation. Hence, if $b\notin \mathrm{cl}(\{a\})$ then $\mathrm{cl}(\{b\})\cap \mathrm{cl}(\{a\})=\emptyset$ and if $b\sim a$ holds then $b\in \mathrm{cl}(\{a\})$ and, hence, $\mathrm{cl}(\{b\})=\mathrm{cl}(\{a\})$ holds. It follows that if g is an automorphism of H and $a\sim b$ holds for some two elements a and b of H, then $g(a)\sim g(b)$ holds as well. Hence, sets of the form $\mathrm{cl}(a)$ are imprimitivity classes of H. Clearly, each of the sets of the form $\mathrm{cl}(a)$ induces a homogeneous substructure of H. By the definition of singly generated, $\mathrm{cl}\,A=\bigcup_{a\in A}\mathrm{cl}(a)$ holds for every finite subset A of H.

(iii) $\Rightarrow$ (i): Assume that A and B are finite subsets of H, with $\mathrm{cl}\,A\cap B=\emptyset$. Let $A'\subseteq A$ be a set of *representatives* of A, that is, for every element $a\in A$, there is an element

$a' \in A'$, with $a \sim a'$, and, for any two different elements $x, y \in A'$, $x \not\sim y$ holds. Observe that then $\mathrm{cl}\, A = \mathrm{cl}\, A'$ holds. Similarly, let B' be a set of representatives of B. Then $\mathrm{cl}\, A' \cap B' = \emptyset$ holds and $\mathrm{cl}\, A' = \bigcup_{a \in A'} \mathrm{cl}(\{a\})$ and $\mathrm{cl}\, B' = \bigcup_{b \in B'} \mathrm{cl}(\{b\})$. So, for all $b \in B'$ and all $a \in A'$, $b \notin \mathrm{cl}(\{a\})$ and, hence, $\mathrm{cl}(\{b\}) \cap \mathrm{cl}(\{a\}) = \emptyset$ and $\mathrm{cl}\, A \cap \mathrm{cl}\, B = \mathrm{cl}\, A' \cap \mathrm{cl}\, B' = \bigcup_{a \in A'} \mathrm{cl}(\{a\}) \cap \bigcup_{b \in B'} \mathrm{cl}(\{b\}) = \emptyset$. $\square$

The following example exhibits a homogeneous structure with binary but infinite language which is transitive and has closure but is not closure-inexhaustible.

Example 1.11. The binary relations of $\mathscr{L}$ are r_n, $n \in \omega$. For the $\mathscr{L}$-structure H, every relation is symmetric, $r_1(x, y)$ implies that $x \neq y$ and H is the connected ω-tree in r_1 (any point has infinitely many neighbors). $r_0(x, y)$ holds iff $x = y$ and, for $n \geqslant 2$, $r_n(x, y)$ holds iff the r_1-path from x to y has length n (here the length of a path is the number of its edges). It is now easy to see that H is vertex-transitive and if $n \geqslant 1$ and $r_n(x, y)$ holds then we denote by $P(x, y)$ the set of inner points of the path from x to y. In other words,

(∗) $z \in P(x, y)$ if there exists an i with $1 \leqslant i \leqslant n-1$ such that $r_i(x, z)$ and $r_{n-i}(z, y)$ hold.

For finite $A \subseteq H$, we define

$$\bar{A} = \{x \mid \exists y, z \in A\ (x \in P(y, z))\} \cup A,$$

$$\delta A = \{x \in A \mid \forall y, z \in A\ (x \notin P(y, z))\}.$$

For $a \in \delta A$,

$$S_A^a = \{x \in H \mid x \notin \bar{A} \wedge P(x, a) \cap \bar{A} = \emptyset\} \cup \{a\}.$$

Observe now that the sets $S_A^a \setminus \{a\}$ for $a \in \delta A$, together with the set $\bar{A}$, form a partition of H. The set $\bar{A}$ is finite and each of the sets S_A^a induces a substructure of H isomorphic to H.

In order to prove that H is homogeneous, we have, according to [3], to prove that if $f: A \to B$ is a local isomorphism for any two finite subsets A and B of H then there exists an automorphism $\bar{f}$ of H which is an extension of f. Let, therefore, $A, B \subseteq H$ be finite and $f: A \to B$ be a local isomorphism. By (∗), the function f can be uniquely extended to an isomorphism $\bar{f}: \bar{A} \to \bar{B}$. If $a \in \delta A$ then $f(a) \in \delta B$; let then $f_a: S_A^a \to S_B^{f(a)}$ be an isomorphism with $f_a(a) = f(a)$. Then isomorphism f_a exists because the automorphism group of H is vertex-transitive and S_A^a is isomorphic to H. From this it follows then that the function $(\bigcup_{a \in \delta A} f_a) \cup \bar{f}$ is an automorphism of H and, hence, H is homogeneous.

Now, clearly, for a finite and nonempty $A \subseteq H$, we get that $\mathrm{cl}\, A = \bar{A}$. Removing $\mathrm{cl}\, A$, the remaining structure fails to be connected and, hence, is not isomorphic to H.

When the language $\mathscr{L}$ is finite, every finite substructure of a model of $\mathscr{L}$ has only finitely many orbits. In the tree H every one-point set has infinitely many different orbits. Therefore, H cannot be described by a finite language.

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